

Periodic Solutions of a Nonlinear Second Order Differential Equation with Delay

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It is proved that the autonomous difference-differential equation

$$\ddot{x}(t) + (a + b) \dot{x}(t) + ab x(t) = -f(x(t-1)) \quad (0)$$

has, for a broad class of functions f , a nonconstant periodic solution whenever the associated characteristic equation has a root with positive real part.

RESULTS AND BACKGROUND

The equation in the abstract is equivalent to the system of differential equations

$$\dot{x}(t) = y(t) - ax(t), \quad (1a)$$

$$\dot{y}(t) = -f(x(t-1)) - by(t). \quad (1b)$$

The object of this paper is to prove the following

THEOREM. *System (1) has a nonconstant periodic solution with a period greater than 2 if the following six conditions hold:*

- (i) a and b are positive constants,
- (ii) $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function,
- (iii) $\xi f(\xi) > 0$ for all $\xi \neq 0$,
- (iv) there is a positive constant κ such that $f(\xi) \geq -\kappa$ for all ξ ,
- (v) f is differentiable at $\xi = 0$,
- (vi) $f'(0) > (a + b)\nu/\sin \nu$,

where the number ν satisfies $0 < \nu < \pi$ and $\cot \nu = (\nu - ab/\nu)/(a + b)$.

The problem of the existence of slowly oscillating periodic solutions has been investigated successfully for the related equations

$$\dot{x}(t) + cx(t) = -f(x(t-1)) \quad (2)$$

and

$$\dot{x}(t) + h(x(t)) \dot{x}(t) = -g(x(t-1)). \quad (3)$$

Equation (2) has been intensively studied in [3, 12, 6, 9], and part of the methods can and will be applied to Eq. (0).

Equation (3) is referred to as the Liénard equation with delay. It seems to be first investigated by Grafton [4], who generalized his results in [5]. Other generalizations are to be found in [11, 7]. However, one crucial assumption is common to all these investigations: The factor $h(x)$ of the first derivative of x has to be negative in a neighborhood of $x = 0$. By way of contrast in Eq. (0) this factor is positive. The considerations regarding the Liénard equation for small x do not apply to Eq. (0). Moreover additional complications arise from the term $abx(t)$ not present in Eq. (3).

The theorem will be derived in accordance with Nussbaum's method [10] by applying the ejective fixed point principle of Browder [2] to an invariant cone in the state space. Unfortunately the cone of monotone functions most often used with this method is not invariant with respect to the flow of Eq. (0). The situation is similar with regard to Eq. (2). Indeed a construction analog to that given in [6] will solve the problem. The theorem will be a consequence of Lemmas 1–6 below. In the last section some applications are indicated.

ANALYSIS

With the abbreviations $\alpha = a + b$, $\beta = ab$, and $\gamma = f'(0)$, the characteristic equation associated with system (1) is

$$\lambda^2 + \alpha\lambda + \beta + \gamma e^{-\lambda} = 0, \quad \lambda \in \mathbb{C}. \quad (4)$$

The first lemma expresses three conditions for the stationary state of system (1) to be unstable.

LEMMA 1. *Let α, β, γ be positive constants. If $\alpha^2 \geq 2\beta$, then the following three conditions are equivalent.*

- (1) *Equation (4) has at least one solution with positive real part.*
- (2) *Equation (4) has precisely one solution λ with $\operatorname{Re} \lambda > 0$ and $0 < \operatorname{Im} \lambda < \pi$.*
- (3) *$\gamma > \alpha\nu_1/\sin \nu_1$, where $0 < \nu_1 < \pi$ and $\cot \nu_1 = (\nu_1 - \beta/\nu_1)/\alpha$.*

Proof. First it will be shown that (3) implies (2).

(a) Assume $\pi^2 + \alpha^2/4 - \beta > 0$. It will be proved that there is a number μ_0 such that for all $s \in [-1, 1]$ the function

$$H_s(\lambda) = \lambda^2 + s\alpha\lambda + s^2\beta + \frac{s+1}{2}\gamma e^{-\lambda} \quad (5)$$

is different from zero on the boundary of the region

$$G_{\tilde{\mu}} = \{\lambda = \mu + i\nu \in \mathbb{C} : 0 < \mu < \tilde{\mu}, -\pi < \nu < \pi\}$$

for each $\tilde{\mu} \geq \mu_0$. Then it follows from Rouché's theorem that $H_1(\lambda)$ has the same algebraic number of zeros in $G_{\tilde{\mu}}$ as $H_{-1}(\lambda)$, which is 2. Since $H_s(\tilde{\lambda}) = \overline{H_s(\bar{\lambda})}$ and $H_1(\mu) \neq 0$ for $\mu \geq 0$, condition (2) holds.

Because of $H_s(\tilde{\lambda}) = \overline{H_s(\bar{\lambda})}$ for all $\lambda \in \mathbb{C}$ the consideration may be restricted to that part of the boundary of $G_{\tilde{\mu}}$ where $\nu \geq 0$. For $\lambda = \mu + i\nu$ the real and imaginary parts of $H_s(\lambda)$ are

$$\operatorname{Re} H_s(\lambda) = \mu^2 - \nu^2 + s\alpha\mu + s^2\beta + \frac{s+1}{2} \gamma e^{-\mu} \cos \nu, \quad (6)$$

$$\operatorname{Im} H_s(\lambda) = 2\nu\mu + s\alpha\nu - \frac{s+1}{2} \gamma e^{-\mu} \sin \nu. \quad (7)$$

$$H_s(\lambda) \neq 0 \quad \text{if } \lambda = i\nu, \quad 0 \leq \nu \leq \pi. \quad (8)$$

Equation (8) holds for $\nu = 0$ because of (6) and $\beta, \gamma > 0$, and for $\nu = \pi$ since $\operatorname{Im} H_s(i\pi) = 0$ implies $s = 0$, but $\operatorname{Re} H_0(i\pi) < 0$.

Let now $0 < \nu < \pi$ and assume $H_s(i\nu) = 0$. $\operatorname{Im} H_{-1}(i\nu) \neq 0$ implies $s \neq -1$. For $s \neq -1$, $\operatorname{Im} H_s(i\nu) = 0$ means $s = (s+1)\gamma \sin \nu / 2\alpha\nu$, therefore $s > 0$. For $s > 0$ Eqs. (6) and (7) and $H_s(i\nu) = 0$ imply

$$\cot \nu = \frac{\nu^2 - s^2\beta}{s\alpha\nu} = h(s, \nu).$$

For fixed $s > 0$ the equation $\cot \nu = h(s, \nu)$ has exactly one solution ν_s in $(0, \pi)$. ν_s increases with respect to s , $\nu_s \leq \nu_1$.

Condition (3) says $\alpha\nu_1 < \gamma \sin \nu_1$. Since \sin is concave in $(0, \pi)$, $\alpha\nu < \gamma \sin \nu$ for $0 < \nu \leq \nu_1$. Therefore $\alpha\nu_s 2s/(s+1) < \gamma \sin \nu_s$ for $0 < s \leq 1$, i.e., $\operatorname{Im} H_s(i\nu) \neq 0$. This proves relation (8).

$$H_s(\lambda) \neq 0 \quad \text{for } \lambda = \mu + i\pi. \quad (9)$$

To prove (9) assume $H_s(\lambda) = 0$. Then it follows from the expression for $\operatorname{Im} H_s(\lambda)$ that $\mu = -s\alpha/2$. This μ evaluated in Eq. (6) results in

$$0 = -s^2 \left(\frac{\alpha^2}{4} - \beta \right) - \pi^2 - \frac{s+1}{2} \gamma e^{-\mu} < 0,$$

a contradiction.

Finally there is a number $\mu_0 > 0$ such that, if $\mu \geq \mu_0$, $|\nu| \leq \pi$, $-1 \leq s \leq 1$, then $\operatorname{Re} H_s(\mu + i\nu) > 0$.

(b) Assume $\beta \geq \pi^2 + \alpha^2/4$. Let $\tilde{\alpha}$ be such that $\tilde{\alpha} > \alpha$ and $\pi^2 + \tilde{\alpha}^2/4 > \beta$. For $s \in [0, 1]$ define $\alpha(s) = \alpha + s(\tilde{\alpha} - \alpha)$, $\gamma(s) = \gamma \cdot (1 + s(\tilde{\alpha}/\alpha - 1))$; note that $\gamma(s)/\alpha(s) = \gamma/\alpha$. Let

$$\tilde{H}_s(\lambda) = \lambda^2 + \alpha(s)\lambda + \beta + \gamma(s)e^{-\lambda}.$$

The assumption $\tilde{H}_s(i\nu) = 0$ for $0 \leq \nu \leq \pi$ is easily disproved for $\nu = 0$ and $\nu = \pi$. For $0 < \nu < \pi$ it follows from $\operatorname{Re} \tilde{H}_s(i\nu) = 0$ and $\operatorname{Im} \tilde{H}_s(i\nu) = 0$ that $\gamma(s) \cos \nu = \nu^2 - \beta$ and $\gamma(s) \cdot \sin \nu = \alpha(s)\nu$, resp. Therefore $\nu = \nu(s)$ is the unique solution of $\cot \nu(s) = (\nu(s)^2 - \beta)/(\alpha(s)\nu(s))$, $\pi/2 < \nu(s) < \pi$; $\nu(s)$ decreases for increasing s .

Therefore and because of condition (3)

$$\frac{\gamma(s)}{\alpha(s)} \sin \nu(s) = \frac{\gamma}{\alpha} \sin \nu(s) \geq \frac{\gamma}{\alpha} \sin \nu(0) > \nu(0) \geq \nu(s), \quad (10)$$

hence $\operatorname{Im} \tilde{H}_s(i\nu(s)) \neq 0$, a contradiction.

For $\lambda = \mu + i\pi$ with $\mu \geq 0$ it is immediately seen that $\operatorname{Im} \tilde{H}_s(\lambda) \neq 0$. Just as in part (a) of this proof it follows that $\tilde{H}_0(\lambda)$ and $\tilde{H}_1(\lambda)$ have the same algebraic number of zeros in the strip $0 < \operatorname{Re} \lambda$, $-\pi < \operatorname{Im} \lambda < \pi$. This number is two for $\tilde{H}_1(\lambda)$ as shown in part (a).

Therefore condition (3) implies condition (2). For $\alpha^2 \geq 2\beta$, according to [1, Chap. 13.8], (1) and (3) are equivalent. This remark completes the proof of Lemma 1. ■

Let $K \subset C([-1, 0]) \times \mathbb{R}$ be the cone defined by

$$K = \{\psi = (\varphi, y_0) : \varphi(-1) = 0, e^{at}\varphi(t) \text{ increasing on } [-1, 0], y_0 \geq 0\}.$$

The following lemma shows that system (1) has oscillatory solutions under even weaker conditions than those in the theorem (note, (vi) implies $f'(0) > a + b$, which implies (11)). Moreover, an operator \mathcal{F} will be defined which maps K into itself. A nontrivial fixed point of this operator corresponds to a nontrivial periodic solution of system (1).

The proof of the lemma exhibits many details about the trajectories (bounds, monotony properties, etc.).

LEMMA 2. *If the conditions (i), (ii), (iii), and (v) of the theorem hold, and if, moreover*

$$f'(0) > ab/(e^{\min(a,b)} - 1), \quad (11)$$

then the solution $(x(t), y(t))$ of system (1) corresponding to an initial condition $\psi \in K - \{0\}$ has the following properties:

(1) *The zeros of $x(t)$ for $t > 0$ form an infinite sequence z_k , $k = 1, 2, \dots$, with $x(z_k) = 0$, $z_{k+1} - z_k > 1$, $\dot{x}(z_{2k-1}) < 0$, $\dot{x}(z_{2k}) > 0$, $y(z_{2k-1}) < 0$, $y(z_{2k}) > 0$, $y(z_{2k-1} + 1) < 0$, $y(z_{2k} + 1) > 0$,*

(2) the function $e^{at}x(t)$ is monotonic increasing on the interval $(z_{2k}, z_{2k} + 1)$ and monotonic decreasing on the interval $(z_{2k-1}, z_{2k-1} + 1)$,

(3) the map $\mathcal{F}: K \rightarrow K$, defined by $\mathcal{F}(0) = 0$ and, for $\psi \neq 0$, $\mathcal{F}(\psi) = (\tilde{\varphi}, \tilde{y}_0)$ with $\tilde{\varphi}(t) = x(t + z_2 + 1)$, $\tilde{y}_0 = y(z_2 + 1)$, satisfies:

For each $M > 0$ there is $\tilde{M} > 0$ such that $\|\psi\| \leq M$ implies $\|\mathcal{F}(\psi)\| \leq \tilde{M}$; $\tilde{M} \rightarrow 0$ as $M \rightarrow 0$.

Proof. The idea is to follow the trajectory $v(t) = (x(t), y(t))$ corresponding to an initial condition $\psi \in K - \{0\}$ along one revolution in the (x, y) -plane.

Let $\|\psi\| = \max(\sup_{t \in [-1, 0]} \varphi(t), y_0) \leq M$. Either $v(0) = (\varphi(0), y_0) \in R_1 = \{(x, y): x \geq 0, y > ax\}$ or $v(0) \in R_2 = \{(x, y): x > 0, 0 \leq y \leq ax\}$. If $v(0) \in R_2$, define $t_1 = 0$. If $v(0) \in R_1$, define $t_1 = \inf\{t \geq 0: v(t) \notin R_1\}$. In this case it follows from Eq. (1a) that x increases for $t \in [0, t_1]$. Since $x = \varphi \geq 0$ on the interval $[-1, 0]$, the function $x(t - 1)$ is nonnegative on $[0, t_1]$. Therefore Eq. (1b) implies $\dot{y}(t) \leq -by(t) < 0$. These observations lead to the estimates

$$0 < x(t) < y(t)/a \leq y_0/a \leq M/a \quad (12)$$

for $t \in (0, t_1)$. Moreover, if t_1 is finite then $v(t_1) \in L_1 = \{(x, y): x > 0, y = ax\} \subset R_2$.

Assume $t_1 > 1$. Then the inequalities

$$\dot{y}(t) \leq -by(y) \leq -bax(t) \leq -bax(1) < 0, \quad t \in [1, t_1)$$

imply t_1 is finite.

In either case $v(t_1) \in R_2$. Let $t_2 = \inf\{t \geq t_1: v(t) \notin R_2 \setminus L_2\}$, where $L_2 = \{(x, y): x > 0, y = 0\}$. In the next step it will be shown that t_2 is finite, $v(t_2) \in L_2$, and the estimate

$$0 \leq y(t) \leq ax(t) \leq M_1 = \max(M, aM) \quad (13)$$

holds for $t \in [t_1, t_2]$.

Since $x(t - 1)$ is nonnegative for $t \in [t_1, t_2]$ it follows from Eqs. (1) that the functions x and y are decreasing as long as $v(t) \in R_2$. Because of assumption (11) there are positive numbers c and δ such that

$$f'(0) > c > ab/(e^{\min(a, b)} - 1) \quad (14)$$

and

$$|f(\xi)| \geq c|\xi| \quad \text{if} \quad |\xi| < \delta. \quad (15)$$

If $x(t) > \delta$ for all $t \in [t_1, t_2]$, then t_2 is finite (otherwise $\dot{y}(t) < -f(x(t - 1)) \leq -\inf\{f(\xi): \xi \in [\delta, M_1]\} < 0$ for all $t \geq t_1 + 1$). Therefore, if $t_2 = \infty$, there is $\tau_1 \geq t_1 + 1$ with $x(t - 1) \leq \delta$ for all $t \geq \tau_1$.

Then it follows from the integral representation of y

$$y(t) = - \int_{t_0}^t f(x(t' - 1)) e^{-b(t-t')} dt' + y(t_0) e^{-b(t-t_0)} \quad (16)$$

with $t_0 = \tau_1$ that for $t \in [\tau_1, \tau_1 + 1]$

$$\begin{aligned} y(t) &\leq y(\tau_1) e^{-b(t-\tau_1)} - c \int_{\tau_1}^t x(t' - 1) e^{-b(t-t')} dt' \\ &\leq ax(\tau_1) e^{-b(t-\tau_1)} - cx(\tau_1) (1 - e^{-b(t-\tau_1)})/b, \end{aligned}$$

and hence $y(\tau_1 + 1) < 0$ (since $c > ab/(e^b - 1)$), which contradicts $t_2 = \infty$. This proves $t_2 < \infty$.

$v(t_2) \in L_2$, if $v(t_2) \notin L_1$ and $v(t_2) \neq 0$. Indeed the trajectory $v(t)$ cannot leave R_2 across the line L_1 , since $\dot{x}(t_2) = 0$ and $\dot{y}(t_2) \leq -by(t_2) < 0$ if $v(t_2) \in L_1$. The case $v(t_2) = 0$ can be excluded as it follows from the integral representation

$$x(t) = \int_{t_0}^t y(t') e^{-a(t-t')} dt' + x(t_0) e^{-a(t-t_0)} \quad (17)$$

with $t_0 = t_1$ that $x(t_2) > 0$.

With $t_0 = t_2$ it follows from (16) that there is a minimal finite number $t_3 \geq t_2$, at which $y(t)$ becomes negative and remains negative at least as long as $x(t - 1) \geq 0$.

Because of (17) with $t_0 = t_2$ the function x decreases in the interval $[t_2, t_3]$ and $x(t_3) > 0$.

Let $z_1 > t_3$ denote the minimal positive number with $x(z_1) = 0$. It follows from Eq. (1a) that $x(t)$ is decreasing in the interval $[t_3, z_1)$.

Assume $z_1 = \infty$.

Case 1. There is $T \geq t_3$ such that $\dot{y}(t) \leq 0$ for all $t \geq T$. Then $\dot{x}(t) \leq y(t) \leq y(T) < 0$ for $t \geq T$. Therefore case 1 implies $z_1 < \infty$.

Case 2. For each $T > t_3$ there is $t^* \geq T$ with $\dot{y}(t^*) > 0$. Because of $\dot{x}(t) \leq -ax(t)$ there is a time $\tau_2 > t_3 + 1$ such that $x(t - 1) < \delta$ for $t \geq \tau_2$ and $\dot{y}(\tau_2) \geq 0$.

For $t \in [\tau_2, \tau_2 + 1]$ the inequality

$$y(t) \leq -cx(\tau_2)/b \quad (18)$$

holds. This is true for all t with $\dot{y}(t) \geq 0$ because of the Eqs. (16) and (15) and the fact that x is decreasing in $[t_3, z_1)$. If $\dot{y}(t) < 0$ then $y(t) \leq y(\tilde{t}) \leq -cx(\tau_2)/b$, where \tilde{t} is the greatest number below t satisfying $\dot{y}(\tilde{t}) = 0$.

Inequality (18) evaluated in Eq. (17) with $t_0 = \tau_2$ and $t = \tau_2 + 1$ results in

$$x(\tau_2 + 1) \leq -cx(\tau_2)(1 - e^{-a})/ab + x(\tau_2)e^{-a} < 0,$$

which contradicts $z_1 = \infty$. Therefore z_1 has to be finite. It follows from Eq. (16) with $t_0 = t_3$ that $y(t) \geq -M_2/b$ for $t \in [t_3, z_1]$ with $M_2 = \max\{f(\xi): 0 \leq \xi \leq M_1\}$.

Altogether we have the estimates

$$0 \leq x(t) \leq M_1, \quad 0 \geq y(t) \geq -M_2/b \quad (19)$$

in the interval $[t_2, z_1]$. Equation (1a) implies $\dot{x}(z_1) = y(z_1) < 0$.

The function $e^{at}x(t)$ is monotonic decreasing in $[z_1, z_1 + 1]$. To see this observe that, according to (17), $e^{at}x(t) = \int_{z_1}^t y(t') e^{at'} dt'$, where the integrand is negative as consequence of Eq. (16) with $t_0 = z_1$. In particular

$$y(z_1 + 1) < 0.$$

From the integral representations of y and x the following bounds are obtainable. If $t \in [z_1, z_1 + 1]$,

$$\begin{aligned} y(t) &\geq y(z_1) - M_2 \int_{z_1}^{z_1+1} e^{-b(z_1+1-t')} dt' \geq -\frac{M_2}{b} (2 - e^{-b}) = -M_3, \\ x(t) &\geq -M_3(1 - e^{-a})/a = -M_4. \end{aligned} \quad (20)$$

Let $\bar{M} = \max\{M_1, M_3, M_4\}$. Clearly $\bar{M} \rightarrow 0$ if $M \rightarrow 0$. Take now as initial conditions for system (1) the pair $\bar{\psi} = (\bar{\varphi}, \bar{y}_0)$ with $\bar{\varphi}(t) = x(z_1 + 1 + t)$, $t \in [-1, 0]$, $\bar{y}_0 = y(z_1 + 1)$. Note that $\bar{\psi} \in -K$ and $0 < \|\bar{\psi}\| \leq \bar{M}$. Then, because of the symmetry properties of system (1), the same reasoning as before shows that the solution $\bar{v} = (\bar{x}, \bar{y})$ of system (1) corresponding to $\bar{\psi}$ satisfies: There is a first zero \bar{z}_1 of \bar{x} , this zero obeys $\bar{y}(\bar{z}_1 + 1) > 0$, $e^{at}\bar{x}(t)$ is monotonic increasing on $[\bar{z}_1, \bar{z}_1 + 1]$, and there are bounds $\bar{M}_1, \dots, \bar{M}_4$ related to \bar{M} in the same way as M_1, \dots, M_4 to M . $z_2 = z_1 + 1 + \bar{z}_1$ is the second zero of $x(t)$, and $\tilde{\varphi}(t) = x(t + z_2 + 1) = \bar{x}(t + \bar{z}_1 + 1)$, $\tilde{y}_0 = y(z_2 + 1) = \bar{y}(\bar{z}_1 + 1)$ have the properties stated in the lemma. Moreover, $\bar{M} = \max\{\bar{M}_1, \bar{M}_3, \bar{M}_4\} \rightarrow 0$ if $M \rightarrow 0$. Repetition of the argument with respect to $\varphi(t) = x(t + z_k + 1)$, $y_0 = y(z_k + 1)$, $k = 2, 3, \dots$, completes the proof. ■

LEMMA 3. Assume the hypothesis of Lemma 2. Then the map \mathcal{F} is continuous and compact on K .

Proof. It follows from Lemma 2(3) that \mathcal{F} is continuous at 0. The map $\psi \mapsto z_2(\psi)$ is continuous on $K - \{0\}$, since the solutions of (1) depend continuously on the initial conditions. Therefore \mathcal{F} is continuous on $K - \{0\}$. To prove the compactness let A be a bounded subset of K , say $\|\psi\| < M$, $\psi \in A$.

Because of Lemma 2(3) the range $\mathcal{F}(A)$ is bounded. Moreover it was shown in the proof of Lemma 2 that the functions $(x(\psi), y(\psi)): [z_2(\psi), z_2(\psi) + 1] \rightarrow \mathbb{R}^2$, with $(x(\psi), y(\psi))$ being the restriction of the solution of (1), corresponding to the initial condition $\psi \in A$, to the interval $[z_2(\psi), z_2(\psi) + 1]$, are uniformly bounded (indeed $\|x(\psi)\|, \|y(\psi)\| \leq \bar{M}$ for all $\psi \in A$). Therefore the set of all functions $x(\psi)$ with $\psi \in A$, being solutions of Eq. (1a), is equicontinuous. The theorem of Arzela–Ascoli implies that $\mathcal{F}(A)$ has compact closure. ■

LEMMA 4. *Assume the hypothesis of Lemma 2 and condition (iv) of the theorem hold. Then the operator \mathcal{F} maps the closed, bounded, and convex set $D = \{(\varphi, y_0) \in K: \|\varphi\| \leq \kappa(1 - e^{-a})/(ab), y_0 \leq \kappa/b\}$ into itself.*

Proof. Let $\psi = (\varphi, y_0) \in D$ and $\mathcal{F}(\varphi, y_0) = (\tilde{\varphi}, \tilde{y}_0)$. It follows from Lemma 2(1) that $y(z_1 + 1) < 0$. Equation (16) with $t_0 = z_1 + 1$ implies

$$y(t) \leq - \int_{z_1+1}^t f(x(t') - 1) e^{-b(t-t')} dt', \quad t \geq z_1 + 1.$$

Since f is bounded below by $-\kappa$ we obtain $y(t) \leq \kappa/b$ for $t \geq z_1 + 1$. In particular $\tilde{y}_0 = y(z_2 + 1) \leq \kappa/b$. Moreover, for $t \in [z_2, z_2 + 1]$ Eq. (17) implies

$$\tilde{\varphi}(t - z_2 - 1) = x(t) = \int_{z_2}^t y(t') e^{-a(t-t')} dt' \leq \frac{\kappa}{b} \frac{1}{a} (1 - e^{-a}). \quad \blacksquare$$

LEMMA 5. *Under the conditions of the theorem, 0 is an ejective fixed point of the operator \mathcal{F} .*

Proof. It will be shown that there is a constant $\rho > 0$ such that for all $\psi \in K - \{0\}$

$$\limsup_{t \rightarrow \infty} |x(t, \psi)| \geq \rho, \quad (21)$$

$x(t, \psi) = x(t)$ being the x -component of the solution belonging to the initial condition ψ . From (21) the ejectivity of 0 follows easily: The bounds for the solutions of (1) estimated in the proof of Lemma 2 show that for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $\|\psi\| < \delta(\epsilon)$ implies $\sup_{t \in [-1, z_2+1]} |x(t, \psi)| < \epsilon$. In particular let $\epsilon = \rho/2$ and $U = \{\psi \in K: \|\psi\| < \delta(\rho/2)\}$.

Because of (21), for each $\psi \in U - \{0\}$ there is some $n \in \mathbb{N}$ with $\mathcal{F}^n \psi \notin U$. This means 0 is ejective.

Proof of (21). System (1) can be written as a single equation

$$\ddot{x}(t) + \alpha \dot{x}(t) + \beta x(t) + \gamma x(t - 1) = h(t), \quad (22)$$

where $h(t) = \gamma x(t - 1) - f(x(t - 1))$.

Let $\lambda = \mu + i\nu$ be the solution of the characteristic equation (4) with $\mu > 0$, $0 < \nu < \pi$.

Then for $T \geq 0$ it follows from (22) by partial integration of $\int_T^x \dot{x}(t) e^{-\lambda t} dt$ and of $\int_T^\infty \dot{x}(t) e^{-\lambda t} dt$ and by afterwards using relation (4) that

$$\int_T^\infty h(t) e^{-\lambda t} dt = -e^{-\lambda T}(\lambda x(T) + \dot{x}(T) + \alpha x(T)) + \gamma \int_{T-1}^T x(t) e^{-\lambda(t+1)} dt.$$

Take the imaginary part of this equation,

$$\operatorname{Im} \left(\int_T^\infty h(t) e^{-\lambda(t-1)} dt \right) = -\nu x(T) - \gamma \int_{T-1}^T x(t) e^{-\mu(t+1-1)} \sin(\nu(t+1-T)) dt. \quad (23)$$

Choose $\epsilon > 0$ such that $\epsilon < \mu \nu e^{-a}/2$.

Then there is $\sigma > 0$ such that

$$|f(\xi) - \gamma \xi| < \epsilon |\xi| \quad \text{for } |\xi| < \sigma. \quad (24)$$

Assume

$$\limsup_{t \rightarrow \infty} |x(t)| < \sigma e^{-a}. \quad (25)$$

Then there is $k_0 \in \mathbb{N}$ with

$$\sup \left\{ |x(t)| : t \in \bigcup_{k=k_0}^\infty [z_k + 1, z_{k+1}] \right\} = \delta < \sigma e^{-a},$$

and there is $n \geq k_0$ with

$$\delta/2 \leq \max\{|x(t)| : t \in [z_n + 1, z_{n+1}]\} = |x(T)| \quad (26)$$

for some $T \in [z_n + 1, z_{n+1}]$.

Since $e^{a(t-z_k)} |x(t)|$ is monotonic increasing on $[z_k, z_k + 1]$, for these t

$$|x(t)| \leq e^a |x(z_k + 1)| \leq e^a \delta < \sigma, \quad \text{if } k \geq k_0.$$

Therefore

$$|x(t)| < e^a \delta < \sigma \quad \text{for all } t \geq z_{k_0}. \quad (27)$$

Using relations (27), (24), (23), and (26) and the fact that $x(T)$ and $x(t)$, $t \in [T-1, T]$, have the same sign the following estimates are obtained

$$\epsilon e^a \delta / \mu \geq \int_T^\infty |h(t)| e^{-\mu(t-T)} dt \geq \left| \operatorname{Im} \left(\int_T^\infty h(t) e^{-\lambda(t-T)} dt \right) \right| \geq |\nu x(T)| \geq \nu \delta / 2.$$

This relation being a contradiction to the choice of ϵ , the assumption (25) is false, hence relation (21) with $\rho = e^{-a}\sigma$ is established. ■

The theorem is an immediate consequence of Lemmas 4, 3, 5, and the following

LEMMA 6 (Browder [2]). *Let D be a closed, bounded, convex set of infinite dimension in a Banach space, and let $\mathcal{F}: D \rightarrow D$ be a continuous and compact map. Then \mathcal{F} has a fixed point which is not ejective.*

APPLICATIONS

Consider two lowpass filters with time constants δ_1 and δ_2 obeying the differential equations

$$\begin{aligned}\delta_1 \dot{y}_1(t) + y_1(t) &= x_1(t), \\ \delta_2 \dot{y}_2(t) + y_2(t) &= x_2(t),\end{aligned}$$

where x_1, x_2 denote the input and y_1, y_2 the output to the first and second filter, respectively. If the output of the first one acts as input to the second one with a delay $\tau_1 \geq 0$, and if the output of the second one is fed back to the first one with a delay $\tau_2 \geq 0$ and with a nonlinear, negative feedback characteristic $(-\tilde{f})$, then y_1 and y_2 obey the system

$$\begin{aligned}\delta_1 \dot{y}_1(t) + y_1(t) &= -\tilde{f}(y_2(t - \tau_2)), \\ \delta_2 \dot{y}_2(t) + y_2(t) &= y_1(t - \tau_1).\end{aligned}\tag{28}$$

Elimination of y_1 leads to the equation

$$\delta_1 \delta_2 \ddot{y}_2(t) + (\delta_1 + \delta_2) \dot{y}_2(t) + y_2(t) = -\tilde{f}(y_2(t - \tau))$$

with the single delay $\tau = \tau_1 + \tau_2$. If $\tau > 0$, system (28) is equivalent to system (1). Note that for $\tau = 0$, if $f = \tilde{f}/(\delta_1 \delta_2)$ satisfies conditions (ii), (iii), (v) of the theorem then the steady state of system (28) is, according to the theorem of Bendixson, always globally asymptotically stable.

System (28) is used for modeling certain biochemical pathways controlling the production of proteins. The delays represent the times needed for the transcription of *DNA* and for the transport of *mRNA*, which code for the protein, from the nucleus to the soma of the cell. For details see [8, 13]. In this last paper, however, the delays are ignored and periodic solutions are obtained only for systems with at least three coupled filters.

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